METHOD TO SOLVE THE NONLINEAR INFINITE HORIZON OPTIMAL CONTROL PROBLEM WITH APPLICATION TO THE TRACK CONTROL OF A MOBILE ROBOT

THOMAS HOLZHÜTER* and THOMAS KLINKE†
Fachbereich Elektrotechnik und Informatik, Hochschule für Angewandte Wissenschaften Hamburg, Berliner Tor 7, 20099 Hamburg, Germany
*holzhueter@etech.haw-hamburg.de
†klinker@etech.haw-hamburg.de

Received October 18, 2005; Revised March 31, 2006

We present a numerical method to solve the infinite time horizon optimal control problem for low dimensional nonlinear systems. Starting from the linear-quadratic approximation close to the origin, the extremal field is efficiently calculated by solving the Euler–Lagrange equations backward in time. The resulting controller is given numerically on an interpolation grid. We use the method to obtain the optimal track controller for a mobile robot. The result is a globally asymptotically stable nonlinear controller, obtained without any specific insight into the system dynamics.

Keywords: Nonlinear optimal control; Euler–Lagrange equations; Hamilton–Jacobi–Bellman equation; track control; mobile robot.

1. Introduction

The nonlinear equivalent of the well-known linear quadratic optimal regulator is still difficult to obtain. One has to face the severe problem to solve the nonlinear Hamilton–Jacobi–Bellman (HJB) equation or alternatively to solve many two-point boundary value problems for the Euler–Lagrange equations, which still is a serious challenge, especially on the infinite time interval. We refer to [Beard et al., 1998; Beeler et al., 2000; Prager, 2000] for some solutions to both problems and their limitations.

In contrast, the numerical method used in this paper, first presented in [Holzhütter, 2004], only requires the solution of initial value problems. It is based on the simple observation that close to the origin the extremal field of the nonlinear system and its optimal cost \( V(x) \) usually can be approximated by those associated with the linearized system, see [Hauser & Osinga, 2001]. The optimal control problem for the linearized system is solved by the well known LQ regulator, see [Anderson & Moore, 1990]. By suitably choosing starting points close to the origin, the state space for the nonlinear system can be sufficiently covered with optimal trajectories. The optimal feedback control law \( u(x) \) then is given on the irregular grid provided by the trajectory points. For illustration, we apply the method to calculate the optimal track control of a mobile robot along a straight line.

The paper is organized as follows. We will shortly recall the nonlinear optimal control problem for affine systems in Sec. 2. In Sec. 3 we describe a method to solve this problem via backward integration of the Euler–Lagrange equations. Section 4
presents the application of the method to calculate the optimal track controller for a mobile robot.

2. The Nonlinear Infinite Time Horizon Optimal Control Problem

We restrict the optimal control problem to the class of single input affine systems

$$\dot{x} = f(x) + g(x)u, \quad x(0) = x_0$$

(1)

with the criterion

$$J = \frac{1}{2} \int_0^\infty (x^T Q x + u^T R u) dt,$$

(2)

where $Q$ is a constant-valued symmetric positive-semidefinite $n \times n$ matrix and $R = 1$ without loss of generality. The vector fields $f, g: \mathbb{R}^n \to \mathbb{R}^n$ are assumed sufficiently smooth and linearly controllable about the equilibrium at the origin, i.e. $f(0) = 0$ and the pair $(A, B) = (Df(0), g(0))$ is controllable, $Df$ denoting the Jacobian matrix of $f$. The optimal control problem consists in finding the feedback control $u(x)$ which minimizes the criterion (2) for all possible initial conditions $x_0$. From the calculus of variations one gets a necessary condition for a control $u$ to minimize the criterion (2), resulting in the three equations

$$\dot{x} = f(x) + g(x)u \quad (3)$$

$$\dot{p} = -(Df(x) + Dg(x)u)^T p - Qx \quad (4)$$

$$u = -R^{-1} g^T(x)p \quad (5)$$

where $p$ denotes the costate vector. Equations (3) and (4) are the well-known Euler–Lagrange or canonical equations. Inserting the algebraic Eq. (5) in the differential Eqs. (3) and (4) one can calculate the extremal field. We note that Eqs. (3) and (4) are also valid if constraints on the control $u$ are present. According to Pontryagin's minimum principle only Eq. (5) has to be replaced by $u = \arg \min_u H(x, p, u)$ where $H = (1/2)(x^T Q x + u^T R u) + p^T (f(x) + g(x)u)$ is the Hamiltonian of the system, see [Kirk, 2004]. Because the Hamiltonian here is a convex function of $u$, this simply leads to the modified control equation

$$u = \text{sat}(-R^{-1} g^T(x)p, \ u_{\text{max}}) \quad (6)$$

where the saturation function $\text{sat}(x, a) = x$ for $|x| \leq a$ and $\text{sat}(x, a) = a \ sign(x)$ for $|x| > a$. We will make use of this generalization in the application in Sec. 4.

3. Calculating the Control Law Via Euler–Lagrange Backward Integration

The approach used in this paper is based on a simple idea. Close to the origin $x = 0$ the system Eq. (1) can be linearized in the form $\dot{x} = Ax + Bu$ with $A = Df(0)$ and $B = g(0)$. For linear systems the solution of the optimal control problem with infinite time horizon is given by the well-known LQ regulator, see [Anderson & Moore, 1990; Kirk, 2004]. The optimal cost for guiding the system from the initial point $x$ to the origin is $V_L(x) = (1/2)x^T S x$ where $S$ is the unique positive definite solution of the algebraic Riccati equation $A^T S + SA - SBR^{-1}B^T S + Q = 0$. Furthermore $p = Sx$ and thus the optimal control law for the LQ regulator is given by $u(x) = -Kx$ with $K = R^{-1}B^T S$.

For the nonlinear system the origin is a saddle point of the Hamiltonian flow in the $2n$-dimensional $(x, p)$-space and the optimal trajectories form its stable manifold $W^S \subset \mathbb{R}^{2n}$. It can be shown, see [Hauser & Osinga, 2001] that close to the origin the Hamiltonian flow of the nonlinear system is locally diffeomorphic to the flow of the linearized system. So instead of solving many complicated two-point boundary value problems we integrate the Euler–Lagrange Eqs. (3) and (4) backward in time starting at points $x_f$ close to the origin. A natural choice for the starting points $x_f$ is to locate them on some level curve $V_L(x) = V_f$ of the optimal cost of the linearized system. In two dimensions this is an ellipse. Because $V_L(x)$ is a Lyapunov function for the linear closed loop system, all optimal trajectories escape from this level curve. The backward integration is continued until a suitable maximum cost level $V_{\text{max}}$ is reached. The main task now is to choose the set of starting points in such a way that the region of interest in state space is covered sufficiently dense with trajectories. We used the following parametrization for the starting points on the ellipse

$$x_f(\xi) = \frac{V_f}{\sqrt{1 - \frac{1}{2} \frac{\ddot{x}_f(\xi)}{\ddot{x}_f(\xi)} S \ddot{x}_f(\xi)}}, \quad \xi \in [0, 2\pi)$$

where $\ddot{x}_f(\xi) = e_1 \cos \xi + e_2 \sin \xi$ and $e_1, e_2$ are the normalized eigenvectors of $S$. Then $\xi$ is the angle of the ray from the origin through $x_f$. We thus get a parametrized family of extremals. Because the optimal cost $V$ along an extremal is monotonically decreasing with increasing time, the state $x$ and the
costate $p$ can be viewed as functions $x(V, \xi), p(V, \xi)$ of the cost $V$ instead of time $t$.

We now integrate the Euler–Lagrange Eqs. (3) and (4) for a set of starting points $x_f(\xi)$, with $\xi \in \{\xi_1, \xi_2, \ldots, \xi_m\} \subset [0, 2\pi)$ until a certain value $V_{\text{max}}$ of the cost is reached and calculate the Euclidean distance in state space between the end points of neighboring trajectories. Between the two trajectories with the largest distance a new extremal with $\xi = (1/2)(\xi_k + \xi_{k+1})$ is inserted where $\xi_k$ and $\xi_{k+1}$ are the parameters of the corresponding neighboring trajectories. This procedure is repeated until a prescribed maximum distance is retained. In this way the optimal feedback control law $u(x)$ is obtained on an irregular grid provided by the points of the calculated extremals. This irregular grid is used to obtain an interpolation of $u(x)$ on a regular grid using triangulation methods, see [O'Rourke, 1994].

It has to be noted that the extremals may intersect, which leads to locations where the optimal cost $V(x)$ is nondifferentiable and the control law $u(x)$ is discontinuous. A special case of this problem occurs, when one state variable is an angle. Then an identification of points mod $2\pi$ (in the corresponding variable) must be considered. In the case of intersecting extremals one obviously always has to follow the trajectory with the lower cost. Thus there may be watershed curves which consist of points with equal costs along different extremals. We used a continuation method to calculate these curves, see [Holzhüter, 2006].

4. Track Control of a Mobile Robot

As a simple example, we apply the method to the track control problem for a mobile robot with two driven wheels. We look at the problem of following a straight line with constant velocity $v = (1/2)(v_l + v_r)$ of the robot center, where $v_l$ and $v_r$ are the speed of the left and right wheels respectively. As control input we use half of the speed difference of the two wheels $u = (1/2)(v_r - v_l)$. The equations of motion in scaled form are simply

$$\begin{align*}
\dot{x}_1 &= \sin x_2 \\
\dot{x}_2 &= u
\end{align*}$$

(7)

where $x_1 = e$ denotes the distance of the robot to the reference line (counted positive to the left), and $x_2 = \theta$ is the angle (counted counter-clockwise) of the robot with the reference line. We note that the state space variable $x_2$ in (7) is an angle so the system is defined on the manifold $\mathbb{R} \times S^1$, where $S^1$ is the unit circle. The criterion has to be adapted to the cylinder topology, see [Holzhüter, 2006] for a more thorough discussion. We used the differential cost $0.005x_1^2 + 0.4\sin^2(x_2/2) + u^2/2$, which is locally quadratic and corresponds to $Q = \text{diag}(0.01, 0.2)$ and $R = 1$ in (2). This results in satisfying closed loop dynamics for our robots with a double pole at $s = -0.316$ for the linearized system.

To demonstrate the power of our approach we also imposed a control constraint $|u| \leq u_{\text{max}} = 0.3$. Figure 1 shows the optimal cost $V(x)$. Close to the
origin the cost levels are elliptic, while more outside they become deformed according to the nonlinearity. The bold black line is the watershed curve, where turning to the left is equally expensive as turning to the right. The control law \( u(x) \) is shown in Fig. 2. Close to the origin the controller is linear. In a large area of state space the controller is saturated. This is visually expressed by choosing a separate color for the saturated range where \( u = \pm 0.3 \). For \( \theta = 0 \) the linear controller would saturate at \( |u| = u_{\text{max}}/K_1 = 3 \), in which \( K = (0.1, 0.63) \) is the LQ gain. This still is approximately valid in the nonlinear case, as can be seen in Fig. 2.

There are some interesting curves in this picture. In contrast to the linear controller the curve \( u = 0 \) is not a straight line but approaches \( \theta = \mp \pi/2 \) as \( e \to \pm \infty \). A roughly linear behavior close to this curve can be observed. The next interesting curves are the two attracting extremals. For large track errors \( e \) these turnpikes also converge to \( \theta = \mp \pi/2 \) as \( e \to \pm \infty \). This can be expected from geometrical insight, because for large track errors the optimal way to approach the reference track is perpendicular towards the track.

The last curves of interest are the two watersheds between turning left or right to approach the

---

**Fig. 2.** Optimal control \( u(x) \) for the track control of a mobile robot. Colors indicate the values of \( u \).

**Fig. 3.** Controlled tracks of a mobile robot using the nonlinear optimal regulator on (left) an interpolation grid and (right) the linear quadratic regulator. For the latter the angle is normalized to \([-\pi, \pi]\).
reference line. Note that, if the vehicle is on the track, i.e. \( e = 0 \), this corresponds to the directions \( \theta = \pm \pi \). However, for large track errors these curves of indecision approach \( \theta = \pm \pi/2 \) as \( e \to \pm \infty \).

For better geometrical understanding, Fig. 3 shows some sample tracks in the \((x, y)\)-driving plane of the mobile robot. The behavior of the nonlinear optimal controller is compared with that of the conventional linear controller. The initial track error \( e \) is varied, while the initial direction is always \( \theta = (3/4)\pi \). It can be seen that the nonlinear controller changes the turning direction somewhere between \( e = 0 \) and \( e = 5 \) see also Fig. 2. The linear controller always has the same turning direction depending on the sign of \( \theta \). Note that the linear controller starts to have bad performance around \( e = 10 \). For track errors above \( e = 23 \) the robot is circling and linear control thus fails.

5. Conclusions
We have presented a numerical procedure to solve the optimal control problem for low dimensional nonlinear systems and applied it to the track control of a mobile robot. The control law was obtained via backward integration of the Euler–Lagrange equations. The method presented here does not make any use of special properties of the considered system. A generalization to higher system dimensions however still has to face serious numerical and conceptual difficulties.

References